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# Closed-form solutions for dynamic analysis of extensional circular Timoshenko beams with general elastic boundary conditions

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**Abstract**

Closed-form solutions for dynamic analysis of extensional circular Timoshenko beams with general elastic boundary conditions are derived. Taking the Laplace transform and some procedures, the system composed of three coupled governing differential equations and six coupled boundary conditions is uncoupled and reduced to a single equation in terms of the angle of rotation due to bending. The explicit relations between the inward radial displacement, the tangential displacement and the angle of rotation due to bending are revealed. Six exact normalized fundamental solutions of the uncoupled governing differential equation are obtained by the Frobenius method. The exact transformed general solution of the uncoupled system is expressed in terms of the six fundamental solutions, using the generalized Green function given by Lin. The systems based on the Rayleigh and Bernoulli–Euler beam theories can be obtained by taking the corresponding limiting procedures. Without the Laplace transform, the exact solutions for the steady and free vibrations of the general system are obtained. The effects of the spring constants, the opening angle, the rotary inertia and the shear deformation on the natural frequencies are investigated. © 2000 Elsevier Science Ltd. All rights reserved.

*Keywords:* Closed-form solution; Circular Timoshenko beam

## 1. Introduction

The vibration theory of curved beams has great importance in many engineering applications such as in the design of machines, bridges and aircraft structures. Several research workers have studied the in-plane vibration of curved beams. An interesting review of the subject can be found in the review articles (Laura and Maurizi, 1987; Chidamparam and Leissa, 1993). So far, many studies have been published on the free vibration of a curved beam with limiting boundary conditions. Little research has been devoted to the forced vibration problem of curved beams.

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Wang and Lee (1974) introduced the dynamic slope–deflection method for the analysis of in-plane forced inextensional vibration of a multispan circular Bernulli–Euler beam with fixed ends. Irie et al. (1980) studied the steady state out-of-plane response of a free-clamped Timoshenko curved beam using the transfer matrix approach. Wang and Issa (1987) studied the steady state in-plane response of a clamped–clamped Timoshenko beam, subjected to a harmonic uniformly distributed load using the dynamic stiffness matrix method. Silva and Urgueira (1988) studied the steady-state out-of-plane response of a free–free Timoshenko curved beam by using the dynamic stiffness matrix method. Wang et al. (1992) studied the steady-state out-of-plane response of a clamped–clamped multispan circular beam subjected to a harmonic uniformly distributed load using the dynamic stiffness matrix method. Huang et al. (1998a,b) derived the in-plane and the out-of-plane transient response of a hinged–hinged and a clamped–clamped non-circular Timoshenko curved beams by using the dynamic stiffness matrix method and the numerical Laplace transform. Till date, there is no study on the dynamic analysis of extensional circular Timoshenko beam with general elastic boundary conditions, subjected to arbitrary external loads and moments.

The purpose of this article is to derive the closed solutions for the dynamic analysis of an extensional circular Timoshenko beam with elastic boundary conditions. The system composed of three coupled governing differential equations and six coupled boundary conditions is transformed into a single equation in terms of the angle of rotation due to bending. The explicit relations between the inward radial displacement, the tangential displacement, and the angle of rotation due to bending are revealed. The six exact fundamental solutions of the uncoupled governing sixth-order differential equation are derived by the Frobenius method. By using the generalized Green's function given by Lin (1998), the exact general solution of the transformed system is derived. Lin obtained the exact solution for the static analysis of extensional circular Timoshenko beams with general elastic nonhomogeneous elastic boundary conditions by using a generalized Green's function of a sixth-order ordinary differential equation with forcing function composed of the delta function and its derivatives. Two systems based on the Rayleigh and the Bernoulli–Euler beam theories are examined by taking the corresponding limiting procedures. The effects of the spring constants, the opening angle, the rotary inertia, and the shear deformation on the natural frequencies of beams are investigated. The stiffness locking phenomena accompanied in the finite element methods do not exist in the proposed method (Lee and Sin, 1994; Yang and Sin, 1995; Bucalem and Bathe, 1995).

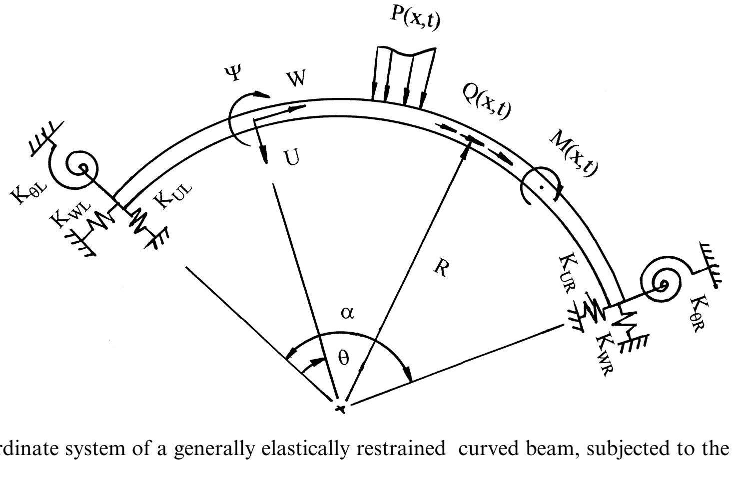


Fig. 1. Geometry and coordinate system of a generally elastically restrained curved beam, subjected to the transverse and axial loads and the external moment.

## 2. Governing equations and boundary conditions

### 2.1. Curved Timoshenko beams

Consider the dynamic response of an elastically restrained extensional circular Timoshenko beam subjected to any transverse forces, axial forces and external moments, as shown in Fig. 1. In terms of the following dimensionless quantities,

$$\begin{aligned}
&m(\xi,\tau)=\frac{M(x,t)L^2}{EI},\quad p(\xi,\tau)=\frac{P(x,t)L^3}{EI},\quad q(\xi,\tau)=\frac{Q(x,t)L^3}{EI},\\
&u=\frac{U}{L},\quad w=\frac{W}{L},\quad \alpha=\frac{L}{R},\\
&\beta_1=\frac{K_{UL}L^3}{EI},\quad \beta_2=\frac{K_{WL}L^3}{EI},\quad \beta_3=\frac{K_{\theta L}L}{EI},\\
&\beta_4=\frac{K_{UR}L^3}{EI},\quad \beta_5=\frac{K_{WR}L^3}{EI},\quad \beta_6=\frac{K_{\theta R}L}{EI},\\
&\mu=\frac{EI}{\kappa GAL^2},\quad \eta=\frac{J}{\rho AL^2},\quad \zeta=\frac{\delta L^2}{EI},\\
&\xi=\frac{R\theta}{L},\quad \tau=\frac{t}{L^2}\sqrt{\frac{EI}{\rho A}},
\end{aligned}\tag{1}$$

the dimensionless governing differential equations and boundary conditions are, respectively,

$$-\frac{1}{\mu}\frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\xi}+\alpha w-\Psi\right)-\zeta\alpha\left(\frac{\partial w}{\partial\xi}-\alpha u\right)+\frac{\partial^2 u}{\partial\tau^2}=p(\xi,\tau),\tag{2}$$

$$\frac{\alpha}{\mu}\left(\frac{\partial u}{\partial\xi}+\alpha w-\Psi\right)-\zeta\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\xi}-\alpha u\right)+\frac{\partial^2 w}{\partial\tau^2}=q(\xi,\tau),\tag{3}$$

$$-\frac{1}{\mu}\left(\frac{\partial u}{\partial\xi}+\alpha w-\Psi\right)-\frac{\partial^2\Psi}{\partial\xi^2}+\eta\frac{\partial^2\Psi}{\partial\tau^2}=m(\xi,\tau),\tag{4}$$

the associated boundary conditions are
at $\xi=0$

$$\gamma_{12}\frac{1}{\mu}\left(\frac{\partial u}{\partial\xi}+\alpha w-\Psi\right)-\gamma_{11}u=0,\tag{5}$$

$$\gamma_{22}\zeta\left(\frac{\partial w}{\partial\xi}-\alpha u\right)-\gamma_{21}w=0,\tag{6}$$

$$\gamma_{32}\frac{\partial\Psi}{\partial\xi}-\gamma_{31}\Psi=0,\tag{7}$$

at $\xi=1$

$$\gamma_{42}\frac{1}{\mu}\left(\frac{\partial u}{\partial\xi}+\alpha w-\Psi\right)+\gamma_{41}u=0,\tag{8}$$

$$\gamma_{52}\zeta\left(\frac{\partial w}{\partial \xi}-\alpha u\right)+\gamma_{51}w=0 \tag{9}$$

$$\gamma_{62}\frac{\partial \Psi}{\partial \xi}+\gamma_{61}\Psi=0, \tag{10}$$

where

$$\gamma_{i1}=\frac{\beta_i}{1+\beta_i},\quad \gamma_{i2}=\frac{1}{1+\beta_i},\qquad i=1,2,3,4,5,6 \tag{11}$$

$W(x,t)$ and $U(x,t)$ are the tangential and the inward radial displacements, respectively, $\Psi(x,t)$ is the angle of rotation due to bending, $t$ is the time variable, $P(x,t)$, $Q(x,t)$ and $M(x,t)$ are the applied distributed transverse loads, axial loads and bending moments, respectively, $E$, $G$, $\kappa$, $I$, $J$ and $A$ denote Young's modulus, shear modulus, shear correction factor, area moment of inertia, mass moment of inertia per unit length and cross-sectional area, respectively, $\rho$ is the density per unit volume; $K_{UL}$, $K_{WL}$ and $K_{\theta L}$ and $K_{UR}$, $K_{WR}$ and $K_{\theta R}$ are the radial translational spring constants, the tangential translational spring constants and the rotational spring constants at $\theta=0$ and $\theta=\alpha$, respectively, $R$ is the radius; $L$ is the length of beam; $\delta=EA$ is the extensional strength. The associated initial conditions are

$$w(\xi,0)=w_0(\xi),\quad \partial w(\xi,0)/\partial\tau=\dot{w}_0(\xi), \tag{12}$$

where $w_0$ and $\dot{w}_0$ are two prescribed initial functions.

It should be noted that if the axial loads $q(\xi,\tau)$ and the moment loads $m(\xi,\tau)$ are neglected and the beam is clamped at both ends, the coupled governing differential equations (2)–(4) and the boundary conditions (5)–(10) become the same as those given by Wang and Issa (1987).

After taking the Laplace transform with respect to time variable $\tau$, the governing differential equations (2)–(4) become

$$-\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}\xi}+\alpha\overline{w}-\overline{\Psi}\right)-\zeta\alpha\left(\frac{\mathrm{d}\overline{w}}{\mathrm{d}\xi}-\alpha\overline{u}\right)+s^2\overline{u}=p^*(\xi,s), \tag{13}$$

$$\frac{\alpha}{\mu}\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}\xi}+\alpha\overline{w}-\overline{\Psi}\right)-\zeta\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\overline{w}}{\mathrm{d}\xi}-\alpha\overline{u}\right)+s^2\overline{w}=q^*(\xi,s), \tag{14}$$

$$-\frac{1}{\mu}\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}\xi}+\alpha\overline{w}-\overline{\Psi}\right)-\frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2}+\eta s^2\overline{\Psi}=m^*(\xi,s), \tag{15}$$

where

$$\begin{aligned}
&\overline{u}(\xi,s)=\int_0^\infty u(\xi,\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau,\qquad \overline{w}(\xi,s)=\int_0^\infty w(\xi,\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau,\\
&\overline{\Psi}(\xi,s)=\int_0^\infty \Psi(\xi,\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau,\qquad \overline{p}(\xi,s)=\int_0^\infty p(\xi,\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau,\\
&\overline{q}(\xi,s)=\int_0^\infty q(\xi,\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau,\qquad \overline{m}(\xi,s)=\int_0^\infty m(\xi,\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau,\\
&m^*(\xi,s)=\overline{m}(\xi,s)+[s\Psi_0(\xi)+\dot{\Psi}_0(\xi)],\qquad p^*(\xi,s)=\overline{p}(\xi,s)+[su_0(\xi)+\dot{u}_0(\xi)],\\
&q^*(\xi,s)=\overline{q}(\xi,s)+[sw_0(\xi)+\dot{w}_0(\xi)].
\end{aligned} \tag{16}$$

Replacing the original dependent variables $\{u, w, \Psi\}$ of Eqs. (5)–(10) by the transformed ones $\{\overline{u}, \overline{w}, \overline{\Psi}\}$, the transformed boundary conditions are obtained.

By substituting Eq. (15) into Eq. (13), one can obtain

$$\overline{u} = \frac{1}{\zeta\alpha^2 + s^2}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(-\frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} + \eta s^2\overline{\Psi} - m^*\right) + \zeta\alpha\frac{\mathrm{d}\overline{w}}{\mathrm{d}\xi} + p^*\right]. \tag{17}$$

Multiplying Eq. (14) and the derivative of Eq. (17) by $\mu/\alpha$ and $\mu(\zeta\alpha^2 + s^2)/\alpha^2$, respectively, and adding the two results yield

$$\overline{\Psi} + \left(\frac{\mu s^2}{\alpha^2} - 1\right)\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}\xi} - \alpha\overline{w}\right) = \frac{\mu}{\alpha^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\eta s^2 - \frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} - m^*\right) + p^*\right] - \frac{\mu}{\alpha}q^*. \tag{18}$$

The tangential displacement in terms of the angle of rotation due to bending is obtained by multiplying Eq. (15) by $\mu(\mu s^2/\alpha^2 - 1)$ and subtracting it from Eq. (18).

$$\begin{aligned}\overline{w} = \frac{\alpha}{2s^2}\Bigg\{&\frac{1}{\alpha^2}\frac{\mathrm{d}^4\overline{\Psi}}{\mathrm{d}\xi^4} + \left[1 - \frac{s^2}{\alpha^2}(\eta + \mu)\right]\frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} + \left[\left(\frac{1}{\alpha^2} - \eta\right)s^2 + \frac{\mu\eta}{\alpha^2}s^4\right]\overline{\Psi} + \frac{1}{\alpha^2}\frac{\mathrm{d}^2 m^*}{\mathrm{d}\xi^2} - \left(\frac{\mu s^2}{\alpha^2} - 1\right)m^* \\ &- \frac{1}{\alpha^2}\frac{\mathrm{d}p^*}{\mathrm{d}\xi} + \frac{1}{\alpha}q^*\Bigg\}.\end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (17), gives the following expression for the inward radial displacement in terms of the angle of rotation due to bending:

$$\begin{aligned}\overline{u} = \frac{1}{\zeta\alpha^2 + s^2}\Bigg\{&\frac{\zeta}{2s^2}\frac{\mathrm{d}^5\overline{\Psi}}{\mathrm{d}\xi^5} - \left[1 + \frac{\zeta}{2}(\eta + \mu) - \frac{\zeta\alpha^2}{2s^2}\right]\frac{\mathrm{d}^3\overline{\Psi}}{\mathrm{d}\xi^3} + \left[\eta s^2 + \frac{\zeta}{2}\left(1 - \alpha^2\eta + \mu\eta s^2\right)\right]\frac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\xi} \\ &+ \frac{\zeta}{2s^2}\frac{\mathrm{d}^3 m^*}{\mathrm{d}\xi^3} + \left(\frac{\zeta\alpha^2}{2s^2} - \frac{\zeta\mu}{2} - 1\right)\frac{\mathrm{d}m^*}{\mathrm{d}\xi} - \frac{\zeta}{2s^2}\frac{\mathrm{d}^2 p^*}{\mathrm{d}\xi^2} + p^* + \frac{\zeta\alpha}{2s^2}\frac{\mathrm{d}q^*}{\mathrm{d}\xi}\Bigg\}.\end{aligned} \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (15), the uncoupled sixth-order governing differential equation in terms of the angle of rotation due to bending is obtained

$$\frac{\mathrm{d}^6\overline{\Psi}}{\mathrm{d}\xi^6} + q_4\frac{\mathrm{d}^4\overline{\Psi}}{\mathrm{d}\xi^4} + q_2\frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} + q_0\overline{\Psi} = \sum_{i=0}^{3}\frac{\mathrm{d}^i p_i(\xi)}{\mathrm{d}\xi^i}, \quad \xi \in (0, 1), \tag{21}$$

where

$$q_4 = 2\alpha^2 - \left(\eta + \mu + \frac{1}{\zeta}\right)s^2,$$

$$q_2 = \left[\mu\eta + \frac{1}{\zeta}(\mu + \eta)\right]s^4 + \left[\alpha^2\left(\mu - 2\eta + \frac{1}{\zeta}\right) + 1\right]s^2 + \alpha^4,$$

$$q_0 = -\left\{\frac{\mu\eta}{\zeta}s^6 + \left[\alpha^2\mu\eta + \frac{1}{\zeta}(1 + \eta)\right]s^4 + (\alpha^2 + \eta\alpha^4)s^2\right\},$$

$$p_3 = p^*,$$

$$p_2 = -\alpha q^* - \left(2\alpha^2 - \mu s^2 - \frac{s^2}{\zeta}\right)m^*,$$

$$p_1 = \left(\alpha^2 - \frac{s^2}{\zeta}\right)p^*,$$

$$p_0 = -\left(\alpha^2 + \frac{s^2}{\zeta}\right)[\alpha q^* + (\alpha^2 + \mu s^2)m^*]. \tag{22}$$

Substituting Eqs. (19) and (20) into the transformed boundary conditions yields the boundary conditions in terms of the angle of rotation due to bending

at $\xi = 0$

$$\gamma_{11} g_1 \frac{\mathrm{d}^5\overline{\Psi}}{\mathrm{d}\xi^5} + \gamma_{11} g_2 \frac{\mathrm{d}^3\overline{\Psi}}{\mathrm{d}\xi^3} + \gamma_{12} g_3 \frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} + \gamma_{11} g_4 \frac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\xi} + \gamma_{12} g_5 \overline{\Psi} = 0, \tag{23}$$

$$\gamma_{22} g_6 \frac{\mathrm{d}^5\overline{\Psi}}{\mathrm{d}\xi^5} + \gamma_{21} g_7 \frac{\mathrm{d}^4\overline{\Psi}}{\mathrm{d}\xi^4} + \gamma_{22} g_8 \frac{\mathrm{d}^3\overline{\Psi}}{\mathrm{d}\xi^3} + \gamma_{21} g_9 \frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} + \gamma_{22} g_{10} \frac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\xi} + \gamma_{21} g_{11} \overline{\Psi} = 0, \tag{24}$$

$$\gamma_{32} \frac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\xi} - \gamma_{31} \overline{\Psi} = 0, \tag{25}$$

at $\xi = 1$

$$\gamma_{41} g_1 \frac{\mathrm{d}^5\overline{\Psi}}{\mathrm{d}\xi^5} + \gamma_{41} g_2 \frac{\mathrm{d}^3\overline{\Psi}}{\mathrm{d}\xi^3} - \gamma_{42} g_3 \frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} + \gamma_{41} g_4 \frac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\xi} - \gamma_{42} g_5 \overline{\Psi} = 0, \tag{26}$$

$$\gamma_{52} g_6 \frac{\mathrm{d}^5\overline{\Psi}}{\mathrm{d}\xi^5} - \gamma_{51} g_7 \frac{\mathrm{d}^4\overline{\Psi}}{\mathrm{d}\xi^4} + \gamma_{52} g_8 \frac{\mathrm{d}^3\overline{\Psi}}{\mathrm{d}\xi^3} - \gamma_{51} g_9 \frac{\mathrm{d}^2\overline{\Psi}}{\mathrm{d}\xi^2} + \gamma_{52} g_{10} \frac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\xi} - \gamma_{51} g_{11} \overline{\Psi} = 0, \tag{27}$$

$$\gamma_{62} \frac{\mathrm{d}\overline{\Psi}}{\mathrm{d}\xi} + \gamma_{61} \overline{\Psi} = 0, \tag{28}$$

where

$$g_1 = 1, \qquad g_2 = \alpha^2 - s^2\left(\mu + \eta + \frac{2}{\zeta}\right), \qquad g_3 = 2s^2\left(\alpha^2 + \frac{s^2}{\zeta}\right),$$

$$g_4 = s^2\left[s^2\left(\frac{2\eta}{\zeta} + \eta\mu\right) + 1 - \eta\alpha^2\right], \qquad g_5 = -2\eta s^4\left(\alpha^2 + \frac{s^2}{\zeta}\right),$$

$$g_6 = 1 - \frac{\alpha^2}{\alpha^2 + s^2/\zeta}, \qquad g_7 = -\frac{1}{\zeta},$$

$$g_8 = \alpha^2 - (\eta + \mu)s^2 + \frac{\alpha^2}{\alpha^2 + s^2/\zeta}\left[\left(\frac{2}{\zeta} + \eta + \mu\right)s^2 - \alpha^2\right],$$

$$g_9 = \frac{1}{\zeta}[(\eta + \mu)s^2 - \alpha^2],$$

$$g_{10} = s^2\left\{1 - \eta\alpha^2 + \mu\eta s^2 - \frac{\alpha^2}{\alpha^2 + s^2/\zeta}\left[\left(\frac{2}{\zeta} + \mu\right)\eta s^2 + 1 - \eta\alpha^2\right]\right\},$$

$$g_{11} = -\frac{s^2}{\zeta}(1 - \eta\alpha^2 + \mu\eta s^2). \tag{29}$$

Letting the dimensionless extensional strength $\zeta \to \infty$, the inextensional curved Timoshenko beam system is obtained. The relation between the tangential and inward radial displacements is obtained from Eqs. (19) and (20)

$$\frac{\partial w}{\partial \xi} = \alpha u, \tag{30}$$

which is well known (Laura et al., 1987).

### 2.2. Curved Rayleigh beams

For Rayleigh beams, the effect of rotatory inertia is considered and that of the shear deformation is neglected. By letting $\mu = 0$, the corresponding uncoupled governing differential equation and boundary conditions can be obtained from Eqs. (21), (22) and (23)–(29), respectively. The inward radial displacement (20) and the tangential displacement (19) in terms of the angle of rotation due to bending are reduced to be, respectively,

$$\begin{aligned}\overline{u} = \frac{1}{\zeta\alpha^2 + s^2}\Bigg\{ & \frac{\zeta}{2s^2}\frac{d^5\overline{\Psi}}{d\xi^5} - \left[1 + \frac{\zeta\eta}{2} - \frac{\zeta\alpha^2}{2s^2}\right]\frac{d^3\overline{\Psi}}{d\xi^3} + \left[\eta s^2 + \frac{\zeta}{2}(1 - \alpha^2\eta)\right]\frac{d\overline{\Psi}}{d\xi} + \frac{\zeta}{2s^2}\frac{d^3 m^*}{d\xi^3} \\ & + \left(\frac{\zeta\alpha^2}{2s^2} - 1\right)\frac{dm^*}{d\xi} - \frac{\zeta}{2s^2}\frac{d^2 p^*}{d\xi^2} + p^* + \frac{\zeta\alpha}{2s^2}\frac{dq^*}{d\xi}\Bigg\},\end{aligned} \tag{31}$$

$$\overline{w} = \frac{\alpha}{2s^2}\left\{\frac{1}{\alpha^2}\frac{d^4\overline{\Psi}}{d\xi^4} + \left[1 - \frac{s^2\eta}{\alpha^2}\right]\frac{d^2\overline{\Psi}}{d\xi^2} + \left(\frac{1}{\alpha^2} - \eta\right)s^2\overline{\Psi} + \frac{1}{\alpha^2}\frac{d^2 m^*}{d\xi^2} + m^* - \frac{1}{\alpha^2}\frac{dp^*}{d\xi} + \frac{1}{\alpha}q^*\right\}. \tag{32}$$

Letting the dimensionless extensional strength $\zeta \to \infty$, the inextensional curved Rayleigh beam system is obtained.

### 2.3. Curved Bernoulli–Euler beams

For Bernoulli–Euler beams, both shear deformation and rotatory inertia are neglected, i.e., $\mu = 0$ and $\eta = 0$. By letting $\mu = 0$ and $\eta = 0$, the corresponding uncoupled governing differential equation and boundary conditions can be obtained from Eqs. (21), (22) and (23)–(29), respectively. The inward radial displacement (20) and the tangential displacement (19) in terms of the angle of rotation due to bending are reduced to be, respectively,

$$\begin{aligned}\overline{u} = \frac{1}{\zeta\alpha^2 + s^2}\Bigg\{ & \frac{\zeta}{2s^2}\frac{d^5\overline{\Psi}}{d\xi^5} - \left(1 - \frac{\zeta\alpha^2}{2s^2}\right)\frac{d^3\overline{\Psi}}{d\xi^3} + \frac{\zeta}{2}\frac{d\overline{\Psi}}{d\xi} + \frac{\zeta}{2s^2}\frac{d^3 m^*}{d\xi^3} + \left(\frac{\zeta\alpha^2}{2s^2} - 1\right)\frac{dm^*}{d\xi} \\ & - \frac{\zeta}{2s^2}\frac{d^2 p^*}{d\xi^2} + p^* + \frac{\zeta\alpha}{2s^2}\frac{dq^*}{d\xi}\Bigg\},\end{aligned} \tag{33}$$

$$\overline{w} = \frac{\alpha}{2s^2}\left\{\frac{1}{\alpha^2}\frac{d^4\overline{\Psi}}{d\xi^4} + \frac{d^2\overline{\Psi}}{d\xi^2} + \frac{s^2}{\alpha^2}\overline{\Psi} + \frac{1}{\alpha^2}\frac{d^2 m^*}{d\xi^2} + m^* - \frac{1}{\alpha^2}\frac{dp^*}{d\xi} + \frac{1}{\alpha}q^*\right\}. \tag{34}$$

Letting the dimensionless extensional strength $\zeta \to \infty$, the inextensional curved Rayleigh beam system is obtained.

## 3. Solution method

### 3.1. General solution

The general solution of the uncoupled governing differential equation (21) can be written as

$$\overline{\Psi}(\xi) = \Psi_p(\xi) + \sum_{i=1}^{6} C_i V_i(\xi), \tag{35}$$

where $\Psi_p(\xi)$ and $\{V_i(\xi)\}$ are the particular solution and the six linearly independent normalized fundamental solutions of Eq. (21), respectively, and $\{C_i\}$ are the constants to be determined. The fundamental solutions satisfy the following normalized condition:

$$\begin{bmatrix} V_1(0) & V_2(0) & V_3(0) & V_4(0) & V_5(0) & V_6(0) \\ V_1'(0) & V_2'(0) & V_3'(0) & V_4'(0) & V_5'(0) & V_6'(0) \\ V_1''(0) & V_2''(0) & V_3''(0) & V_4''(0) & V_5''(0) & V_6''(0) \\ V_1'''(0) & V_2'''(0) & V_3'''(0) & V_4'''(0) & V_5'''(0) & V_6'''(0) \\ V_1^{(4)}(0) & V_2^{(4)}(0) & V_3^{(4)}(0) & V_4^{(4)}(0) & V_5^{(4)}(0) & V_6^{(4)}(0) \\ V_1^{(5)}(0) & V_2^{(5)}(0) & V_3^{(5)}(0) & V_4^{(5)}(0) & V_5^{(5)}(0) & V_6^{(5)}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{36}$$

Substituting the general solution (35) into the boundary conditions (23)–(29), the associated coefficients can be obtained via the following relation:

$$\begin{bmatrix} \delta_{36} & \delta_{35} & \delta_{34} & \delta_{33} & \delta_{32} & \delta_{31} \\ \delta_{26} & \delta_{25} & \delta_{24} & \delta_{23} & \delta_{22} & \delta_{21} \\ \delta_{16} & \delta_{15} & \delta_{14} & \delta_{13} & \delta_{12} & \delta_{11} \\ \gamma_{11}g_1 & 0 & \gamma_{11}g_2 & \gamma_{12}g_3 & \gamma_{11}g_4 & \gamma_{12}g_5 \\ \gamma_{22}g_6 & \gamma_{21}g_7 & \gamma_{22}g_8 & \gamma_{21}g_9 & \gamma_{22}g_{10} & \gamma_{21}g_{11} \\ 0 & 0 & 0 & 0 & \gamma_{32} & -\gamma_{31} \end{bmatrix} \begin{bmatrix} C_6 \\ C_5 \\ C_4 \\ C_3 \\ C_2 \\ C_1 \end{bmatrix} = \begin{bmatrix} A_6 \\ A_5 \\ A_4 \\ A_3 \\ A_2 \\ A_1 \end{bmatrix}, \tag{37}$$

where

$$A_1 = -\gamma_{32}\Psi_p'(0) + \gamma_{31}\Psi_p(0),$$

$$A_2 = -(\gamma_{22}g_6\Psi_p^{(5)}(0) + \gamma_{21}g_7\Psi_p^{(4)}(0) + \gamma_{22}g_8\Psi_p'''(0) + \gamma_{21}g_9\Psi_p''(0) + \gamma_{22}g_{10}\Psi_p'(0) + \gamma_{21}g_{11}\Psi_p(0)),$$

$$A_3 = -(\gamma_{11}g_1\Psi_p^{(5)}(0) + \gamma_{11}g_2\Psi_p'''(0) + \gamma_{12}g_3\Psi_p''(0) + \gamma_{11}g_4\Psi_p'(0) + \gamma_{12}g_5\Psi_p(0)),$$

$$A_4 = \gamma_{41}g_1\Psi_p^{(5)}(1) + \gamma_{41}g_2\Psi_p'''(1) - \gamma_{42}g_3\Psi_p''(1) + \gamma_{41}g_4\Psi_p'(1) - \gamma_{42}g_5\Psi_p(1),$$

$$A_5 = -\gamma_{52}g_6\Psi_p^{(5)}(1) + \gamma_{51}g_7\Psi_p^{(4)}(1) - \gamma_{52}g_8\Psi_p''(1) + \gamma_{51}g_9\Psi_p''(1) - \gamma_{52}g_{10}\Psi_p'(1) + \gamma_{51}g_{11}\Psi_p(1), \tag{38}$$

$$A_6 = -\gamma_{62}\Psi_p'(1) - \gamma_{61}\Psi_p(1),$$

$$\delta_{1i} = -(\gamma_{41}g_1V_i^{(5)}(1) + \gamma_{41}g_2V_i'''(1) - \gamma_{42}g_3V_i''(1) + \gamma_{41}g_4V_i'(1) - \gamma_{42}g_5V_i(1)),$$

$$\delta_{2i} = -(-\gamma_{52}g_6V_i^{(5)}(1) + \gamma_{51}g_7V_i^{(4)}(1) - \gamma_{52}g_8V_i'''(1) + \gamma_{51}g_9V_i''(1) - \gamma_{52}g_{10}V_i'(1) + \gamma_{51}g_{11}V_i(1)),$$

$$\delta_{3i} = \gamma_{62}V_i'(1) + \gamma_{61}V_i(1), \quad i = 1, 2, 3, 4, 5, 6.$$

By taking the inverse Laplace transform, one obtains the transient response of the system

$$\psi(\xi,\tau) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \overline{\Psi}(\xi,s)\mathrm{e}^{s\tau}\,\mathrm{d}s, \quad \xi \in (0,1), \tag{39}$$

where $j^2 = -1$ and $c$ is the constant associated with the inverse Laplace transform (Sneddon, 1972).

### 3.2. Normalized homogeneous solution

A power series representation of the fundamental solutions can be constructed by the Frobenius method (Lee and Lin, 1992). One can assume that the six independent fundamental solutions $V_i(\xi)$ of Eq. (21) are in the form of

$$V_i = \sum_{n=0}^{\infty} k_{i,n}\xi^n, \quad i = 1,2,\ldots,6, \tag{40}$$

$$\begin{aligned}
&\text{for } V_1(\xi): \quad k_{1,0} = 1, \quad k_{1,1} = k_{1,2} = k_{1,3} = k_{1,4} = k_{1,5} = 0,\\
&\text{for } V_2(\xi): \quad k_{2,1} = 1, \quad k_{2,0} = k_{2,2} = k_{2,3} = k_{2,4} = k_{2,5} = 0,\\
&\text{for } V_3(\xi): \quad k_{3,2} = \tfrac{1}{2}, \quad k_{3,0} = k_{3,1} = k_{3,3} = k_{3,4} = k_{3,5} = 0,\\
&\text{for } V_4(\xi): \quad k_{4,3} = \tfrac{1}{6}, \quad k_{4,0} = k_{4,1} = k_{4,2} = k_{4,4} = k_{4,5} = 0,\\
&\text{for } V_5(\xi): \quad k_{5,4} = \tfrac{1}{24}, \quad k_{5,0} = k_{5,1} = k_{5,2} = k_{5,3} = k_{5,5} = 0,\\
&\text{for } V_6(\xi): \quad k_{6,5} = \tfrac{1}{120}, \quad k_{6,0} = k_{6,1} = k_{6,2} = k_{6,3} = k_{6,4} = 0.
\end{aligned} \tag{41}$$

These fundamental solutions satisfy the normalization condition (36). Upon substituting Eq. (40) into Eq. (21) and collecting the coefficients of like powers of $\xi$, the following recurrence formula can be obtained:

$$k_{i,\ell+6} = \frac{(\ell+4)(\ell+3)(\ell+2)(\ell+1)q_4 k_{i,\ell+4} + (\ell+2)(\ell+1)q_2 k_{i,\ell+2} + q_0 k_{i,\ell}}{(\ell+6)(\ell+5)(\ell+4)(\ell+3)(\ell+2)(\ell+1)}, \quad \ell = 0,1,2,\ldots \tag{42}$$

With this recurrence formula, one can generate the six exact normalized fundamental solutions of Eq. (21).

### 3.3. Particular solution

The Green function of an $n$th-order ordinary differential equation with constant coefficients obtained by Lin (1998) is applied to derive the particular solution of Eq. (21). The particular solution $\Psi_p(\xi)$ expressed in terms of the normalized fundamental solutions $V_i(\xi)$ can be obtained:

$$\Psi_p(\xi) = \sum_{i=0}^{3}\int_0^1 p_i(x)E_i(\xi - x)\,\mathrm{d}x, \tag{43}$$

where

$$\begin{aligned}
E_0(\xi) &= V_6(\xi)H(\xi),\\
E_1(\xi) &= V_5(\xi)H(\xi),\\
E_2(\xi) &= [V_4(\xi) - q_4V_6(\xi)]H(\xi),\\
E_3(\xi) &= [V_3(\xi) - q_4V_5(\xi)]H(\xi)
\end{aligned} \tag{44}$$

in which $H(\xi)$ is the Heaviside function.

## 4. Steady motion and free vibration

Consider the steady motion of a curved Timoshenko beam, subjected to harmonic excitations. The loads are assumed to be in the form of

$$m(\zeta,\tau) = \tilde{m}(\xi)\sin\varpi\tau, \quad p(\xi,\tau) = \tilde{p}(\xi)\sin\varpi\tau, \quad q(\zeta,\tau) = \tilde{q}(\xi)\sin\varpi\tau, \tag{45}$$

where $\varpi$ is the dimensionless frequency of excitation, i.e., $\varpi = \overline{\Omega}L^2\sqrt{\rho A/EI}$ in which $\overline{\Omega}$ is the physical frequency of excitation; then, the displacements and the angle of rotation due to bending can be written as, respectively,

$$u(\xi,\tau) = \tilde{u}(\xi)\sin\varpi\tau, \quad w(\xi,\tau) = \tilde{w}(\xi)\sin\varpi\tau, \quad \Psi(\xi,\tau) = \tilde{\Psi}(\xi)\sin\varpi\tau. \tag{46}$$

Substituting Eqs. (45) and (46) into Eqs. (2)–(11), the corresponding governing ordinary differential equations and the associated boundary conditions in terms of $\{\tilde{m},\tilde{p},\tilde{q},\tilde{u},\tilde{w},\tilde{\Psi}\}$ are obtained. Replacing the parameters $\{s,m^*,p^*,q^*,\overline{u},\overline{w},\overline{\Psi}\}$ of the transformed governing Eqs. (13)–(15) of the transient motion by $\{j\varpi,\tilde{m},\tilde{p},\tilde{q},\tilde{u},\tilde{w},\tilde{\Psi}\}$, Eqs. (13)–(15) become the governing ordinary differential equations of the steady motion. Similarly, the associated explicit relations, the uncoupled governing differential equation, the boundary conditions and the steady responses can be obtained by replacing the parameters $\{s,m^*,p^*,q^*,\overline{u},\overline{w},\overline{\Psi}\}$ in Eqs. (19)–(29) and (35)–(38) by $\{j\varpi,\tilde{m},\tilde{p},\tilde{q},\tilde{u},\tilde{w},\tilde{\Psi}\}$.

Consider the free vibration of the system. The Laplace transform parameter $s$ of the forced vibration system is replaced by $j\omega$ where $\omega$ is the dimensionless angular natural frequency, i.e., $\omega = \Omega L^2\sqrt{\rho A/EI}$ in which $\Omega$ is the physical angular natural frequency. Because the coefficients $A_i$ of Eq. (37) are zero, the determinant of the first square matrix in Eq. (37) is zero. Then, the frequency equation of the curved beam can be obtained

$$\begin{vmatrix} \delta_{36} & \delta_{35} & \delta_{34} & \delta_{33} & \delta_{32} & \delta_{31} \\ \delta_{26} & \delta_{25} & \delta_{24} & \delta_{23} & \delta_{22} & \delta_{21} \\ \delta_{16} & \delta_{15} & \delta_{14} & \delta_{13} & \delta_{12} & \delta_{11} \\ \gamma_{11}g_1 & 0 & \gamma_{11}g_2 & \gamma_{12}g_3 & \gamma_{11}g_4 & \gamma_{12}g_5 \\ \gamma_{22}g_6 & \gamma_{21}g_7 & \gamma_{22}g_8 & \gamma_{21}g_9 & \gamma_{22}g_{10} & \gamma_{21}g_{11} \\ 0 & 0 & 0 & 0 & \gamma_{32} & -\gamma_{31} \end{vmatrix} = 0. \tag{47}$$

The roots of the frequency equation are the natural frequencies of the system.

## 5. Verification and discussion

The following examples are given to illustrate the validity and the accuracy of the analysis and study the dynamic behavior of a curved beam.

*Example 1*: Consider the free vibration of Timoshenko curved beams. The comparison of the presented frequencies of the beams to those given by Wolf (1971) and Tufekci and Arpaci (1998) is made. Table 1 shows that the presented numerical results and those given by Tufekci and Arpaci (1998) are very consistent. Tufekci and Arpaci (1998) derived the closed-form solutions for the free in-plane vibration of an extensional circular Timoshenko beam with some limiting boundary conditions by the fundamental matrix method.

*Example 2*: Figs. 2–4 show the influence of the radial spring constant $\beta_4$, the tangential spring constant $\beta_5$, and the rotational spring constant $\beta_6$ on the first four natural frequencies of the curved beams clamped at $\xi = 0$. The solid and the dashed lines denote the natural frequencies of a curved beam with the opening angles $\alpha$ of 180° and 60°, respectively. Fig. 2a and b shows that when the constants $\beta_5$ and $\beta_6$ are zero and the constant $\beta_4$ is decreased to a certain value, the natural frequencies will approach constant values. This means that the spring constant is seen to be zero and the beam is free at $\xi = 1$. Moreover, when the spring constant is increased to a certain value, the natural frequencies approach constant values. This means that

Table 1
The dimensionless frequencies of Timoshenko curved rectangular beams $[1/(\alpha\sqrt{\eta}) = 100,\ v = 0.3,\ \kappa = 5/6,\ \zeta = 1/\eta,\ \mu = 2(1+v)\eta/\kappa]$

| $\alpha$ | Mode | Hinged–hinged | | | Clamped–clamped | | |
|---|---|---|---|---|---|---|---|
| | | Wolf (1971) | Tufekci and Arpaci (1998) | Present study | Wolf (1971) | Tufekci and Arpaci (1998) | Present study |
| 150° | 1 | 26.43 | 26.4079 | 26.4079 | 47.66 | 47.5326 | 47.5326 |
| | 2 | 72.71 | 72.5587 | 72.5588 | 99.32 | 98.8691 | 98.8697 |
| | 3 | 143.1 | 142.5925 | 142.5931 | 182.4 | 181.2108 | 181.2114 |
| | 4 | 229.2 | 227.9351 | 227.9352 | 274.0 | 271.5375 | 271.5344 |
| | 5 | 339.2 | 336.4950 | 336.4755 | 396.8 | 391.9823 | 391.9637 |
| 180° | 1 | 22.37 | 22.3497 | 22.3497 | 43.25 | 43.1709 | 43.1709 |
| | 2 | 68.27 | 68.1644 | 68.1644 | 95.06 | 94.7557 | 94.7559 |
| | 3 | 137.8 | 137.4288 | 137.4288 | 176.5 | 175.7111 | 175.7105 |
| | 4 | 224.6 | 223.7427 | 223.7416 | 270.2 | 268.4875 | 268.4864 |
| | 5 | 334.0 | 332.0705 | 332.0712 | 391.1 | 387.7377 | 387.7465 |

the spring constant is seen to be infinite and the beam is radially hinged at $\xi = 1$. When the rotary inertia and the ratio between bending and shear rigidities are large, the influence of the opening angle $\alpha$ on the natural frequencies of higher modes is greater than on those of lower modes.

Fig. 3a and b shows the influence of the tangential spring constant $\beta_5$ on the frequencies of a beam clamped at $\xi = 0$ and radially hinged at $\xi = 1$. When the rotary inertia and the ratio between bending and the shear rigidities are large, the influence of the tangential spring constant $\beta_5$ on the second and third frequencies of a beam with the opening angle $\alpha$ of 60° and the fourth frequencies of a beam with the opening angle $\alpha$ of 180° is small. The reason is that the corresponding mode shapes are dominant in the inward radial displacement. Fig. 3b shows that when the rotary inertia and the ratio between bending and shear rigidities are small, the influence of the spring constant $\beta_5$ on the natural frequencies of higher modes

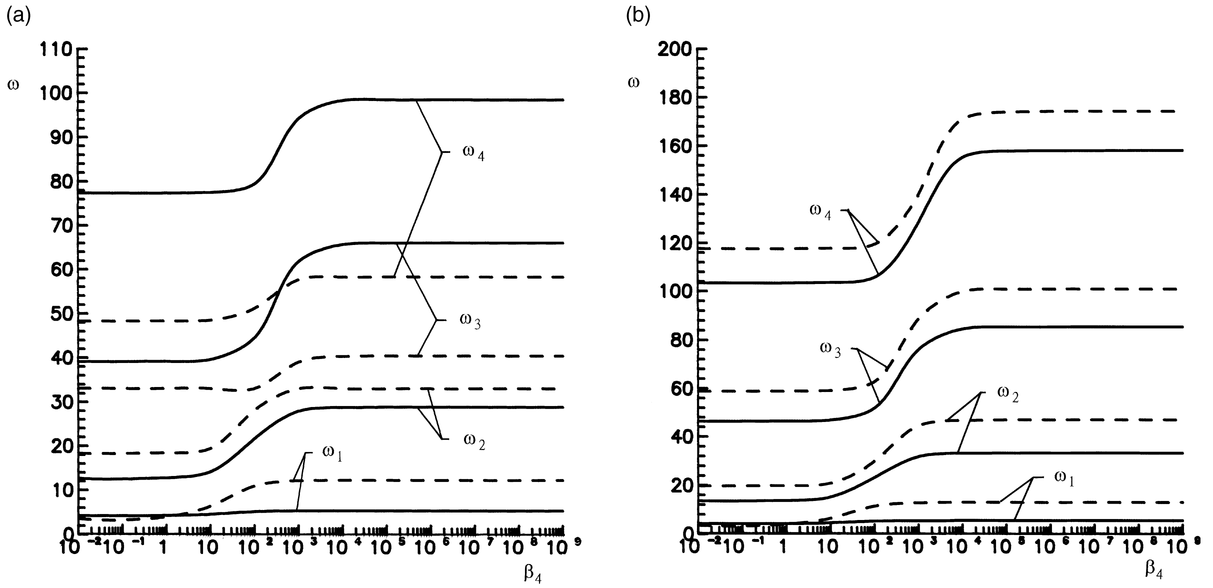


Fig. 2. The influence of the translational spring constant $\beta_4$ on the first four frequencies of a cantilever curved beam: (a) $\beta_5 = \beta_6 = 0$, $\eta = 0.001$, $\zeta = 1000$, $\mu = 0.00312$; (——) $\alpha = 180°$; (- - -) $\alpha = 60°$. (b) $\beta_5 = \beta_6 = 0$, $\eta = 0.00001$, $\zeta = 100,000$, $\mu = 0.0000312$; (——) $\alpha = 180°$; (- - -) $\alpha = 60°$.

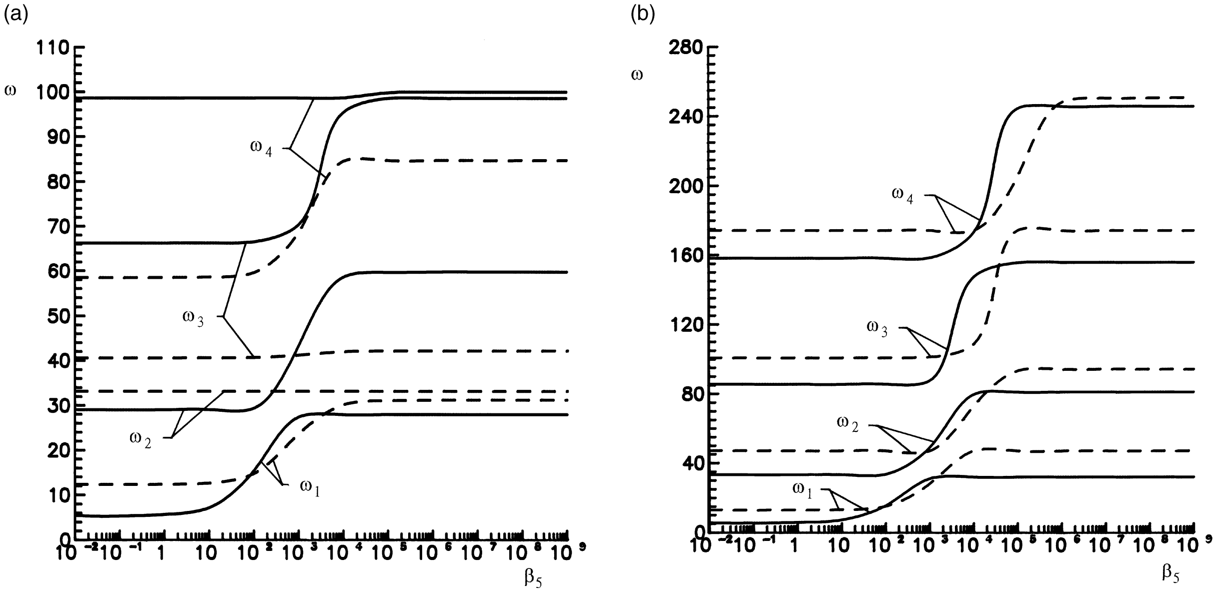


Fig. 3. The influence of the translational spring constant $\beta_5$ on the first four frequencies of a curved beam clamped at $\xi = 0$: (a) $\beta_4 \to \infty$, $\beta_6 = 0$, $\eta = 0.001$, $\zeta = 1000$, $\mu = 0.00312$; (——) $\alpha = 180°$; (- - -) $\alpha = 60°$. (b) $\beta_4 \to \infty$, $\beta_6 = 0$, $\eta = 0.00001$, $\zeta = 100,000$, $\mu = 0.0000312$; (——) $\alpha = 180°$; (- - -) $\alpha = 60°$.

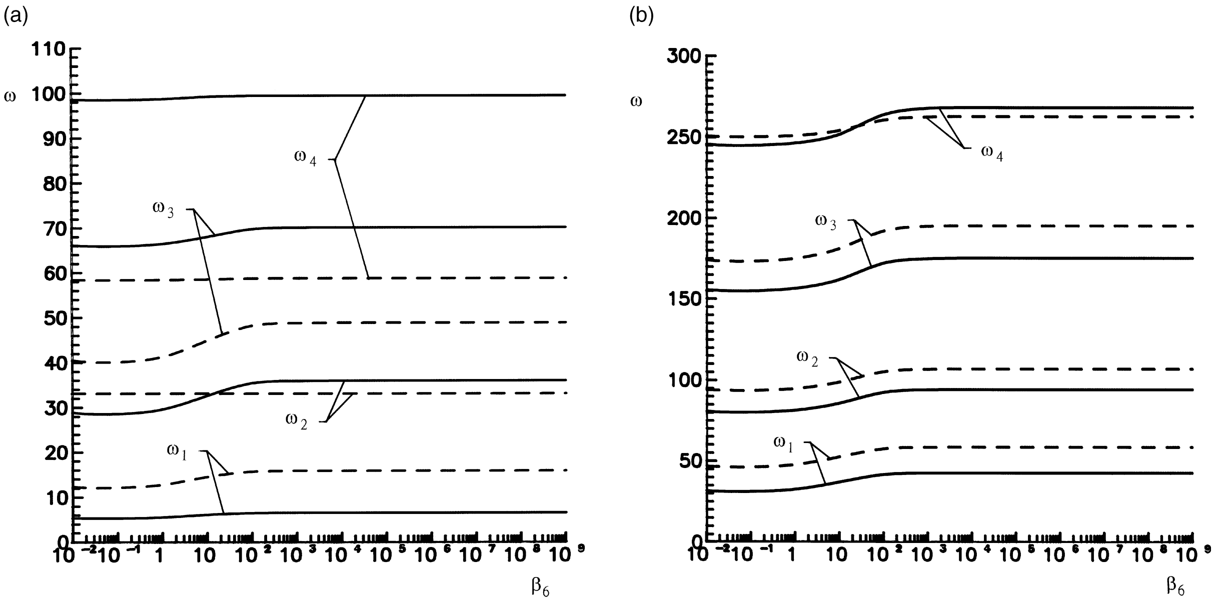


Fig. 4. The influence of the rotational spring constant $\beta_6$ on the first four frequencies of a curved beam clamped at $\xi = 0$: (a) $\beta_4$ and $\beta_5 \to \infty$, $\eta = 0.001$, $\zeta = 1000$, $\mu = 0.00312$; (——) $\alpha = 180°$; (- - -) $\alpha = 60°$. (b) $\beta_4$ and $\beta_5 \to \infty$, $\eta = 0.00001$, $\zeta = 100,000$, $\mu = 0.0000312$; (——) $\alpha = 180°$; (- - -) $\alpha = 60°$.

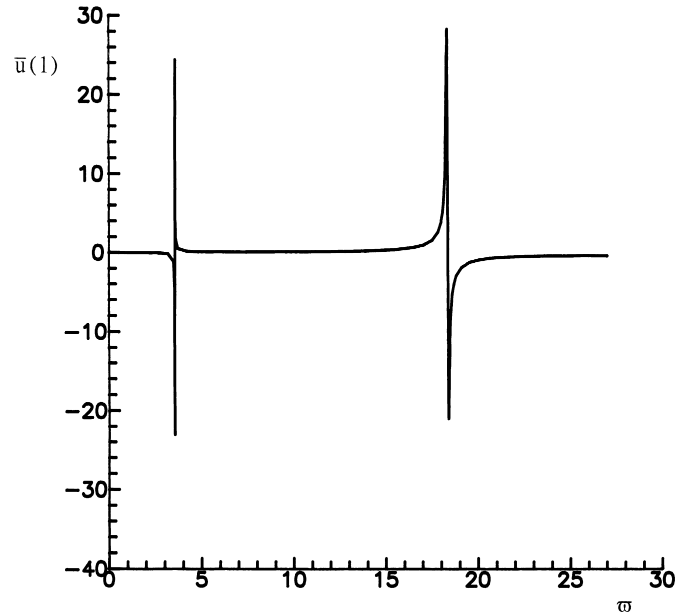


Fig. 5. The amplitudes of steady response at the right end of the cantilever curved beam subjected to a harmonic transverse load at $\xi = 0.5$, $\eta = 0.001$, $\zeta = 1000$, $\mu = 0.00312$; (——) $\alpha = 180°$; (- - -) $\alpha = 60°$.

is greater than on those of lower modes. Fig. 4a and b shows that the influence of the rotational spring constant $\beta_6$ on the natural frequencies of the beams hinged at $\xi = 1$, i.e., $\beta_4$ and $\beta_5 \to \infty$, is small.

*Example 3*: In Fig. 5, the vibrational response curve at the tip of the cantilever curved beam is illustrated. It shows that when the transverse harmonic excitation frequencies approach the natural frequencies of the beam, the response increases rapidly and becomes infinite as the transverse harmonic excitation frequencies coincide with the natural frequencies.

## 6. Conclusion

In this article, the closed-form solutions for dynamic analysis of extensional circular Timoshenko beams with general elastic boundary conditions are obtained. The explicit relations between the inward radial displacement, the tangential displacement and the angle of rotation due to bending are revealed. The uncoupled equation expressed in terms of the angle of rotation due to bending is obtained. The six exact normalized fundamental solutions of the characteristic governing differential equation is obtained by the Frobenius method. The general solution of the system in terms of the six fundamental solutions is obtained by using the generalized Green function given by Lin (1998). Two systems based on the Rayleigh and Bernoulli–Euler beam theories are easily obtained by taking the corresponding limiting conditions. Moreover, the exact solutions for the steady and free vibrations of the systems are obtained.

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